Problem Set 4 due October 7, at 10 AM, on Gradescope (via Stellar)

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue

Problem 1: We will to reconstruct a matrix A such that the general solution to the equation:

$$A \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \qquad \text{is} \qquad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(1) How many rows and columns does A have?   *(5 points)*

(2) Based on the information in the equation above, what is the second column of A?   *(5 points)*

(3) Find the entire matrix A.   *(10 points)*

Problem 2: An $m \times n$ rank 1 matrix has the property that all its columns are multiples of each other, so they are of the form:
$$A = \begin{bmatrix} a_1\boldsymbol{b} & | & a_2\boldsymbol{b} & | & ... & | & a_n\boldsymbol{b} \end{bmatrix}$$
for some non-zero vector $\boldsymbol{b} = \begin{bmatrix} b_1 \\ ... \\ b_m \end{bmatrix}$ and some scalars $a_1, ..., a_n$, not all 0.

(1) Write A as the product of a $m \times 1$ matrix with an $1 \times n$ matrix.   *(5 points)*

(2) Use part 1 to obtain a simple formula for the symmetric matrix $A^T A$ in terms of the a_i's and b_j's? *Hint: if you're stuck, try the 3×2 case for some intuition.*   *(10 points)*

(3) What is the rank of $A^T A$ from part (2)?   *(5 points)*

Problem 3: Find a basis for the vector space spanned by the vectors:

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 2 \\ -3 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ -2 \\ 1 \\ 4 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ -4 \\ 3 \\ 6 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 4 \\ -6 \\ 3 \end{bmatrix}$$

Explain your method.   *(20 points)*

Problem 4: For any two subspaces $V, W \subset \mathbb{R}^m$, we will write $V + W$ for the subspace consisting of all vectors of the form $\boldsymbol{v} + \boldsymbol{w}$, for arbitrary $\boldsymbol{v} \in V$ and $\boldsymbol{w} \in W$.

(1) If $\boldsymbol{v}_1, ..., \boldsymbol{v}_k$ is a basis of V and $\boldsymbol{w}_1, ..., \boldsymbol{w}_l$ is a basis of W, explain why the set $\boldsymbol{v}_1, ..., \boldsymbol{v}_k, \boldsymbol{w}_1, ..., \boldsymbol{w}_l$ spans the subspace $V + W$. *(5 points)*

(2) Show that the dimension of $V + W$ is $\leq$ the sum of dimensions of V and W. Give an example of when the inequality is strict (i.e. $\dim V + W < \dim V + \dim W$). *(5 points)*

(3) Given two matrices A and B of the same shape, what is the relation between $C(A + B)$ and the subspaces $C(A)$ and $C(B)$? *(5 points)*

(4) Use the previous parts to show that $\mathrm{rank}\,(A + B) \leq \mathrm{rank}\,(A) + \mathrm{rank}\,B$. *(5 points)*

Problem 5: do problem 31 in Section 3.5 of the textbook. Make sure you explicitly describe the four subspaces for the matrix A therein. *(20 points)*